No Complete Problem for Constant-Cost Randomized Communication

Yuting Fang
Ohio State University

Lianna Hambardzumyan
Hebrew University of Jerusalem

Nathaniel Harms
EPFL

Pooya Hatami
Ohio State University

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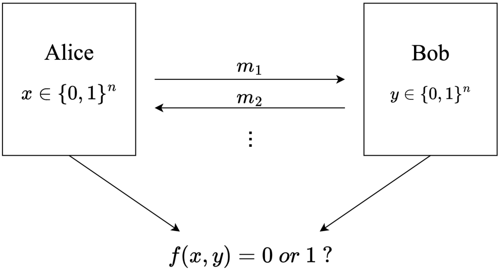
Outline

1. Communication Complexity
   - Models and the Constant-Cost Randomized class BPP^0
2. Landscape of BPP^0
   - A infinite k-Hamming Distance Hierarchy
   - No complete problem
3. Proof Sketch
4. Open Problems

Communication Complexity

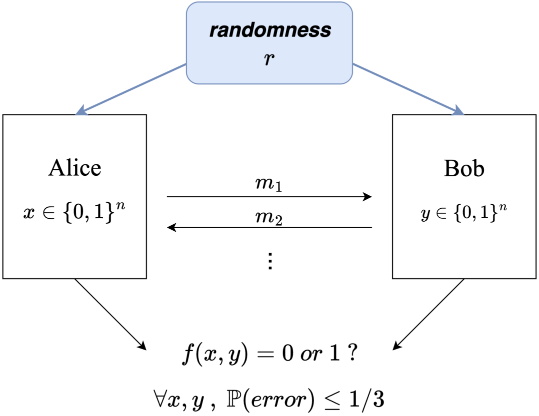
Deterministic model



The cost of protocol is the number of bits exchanged.

Communication Complexity

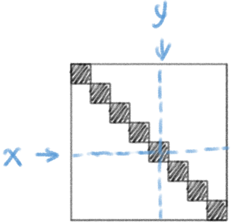
(Public-coin, bounded-error) Randomized model



Communication Complexity

Randomized models are more powerful!

Example: EQUALITY problem



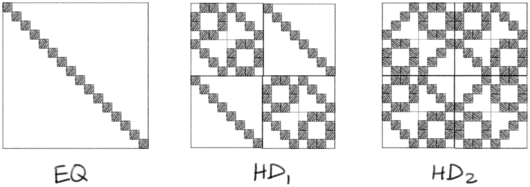
Deterministic: n bits

Randomized: $\mathbf{O(1)}$ bits, by hashing

The k-Hamming Distance Problem

Generalization of EQ

$$\mathrm{HD}_k(x,y) := 1 \iff \mathrm{dist}(x,y) = k$$



$\mathrm{R}(\mathrm{EHD}_k) = \Theta(k \log k)$ [Yao03, HSZZ06, Sag18]

$\Rightarrow$ when k is constant, HD_k admits **constant**-cost randomized protocols.

The Class BPP^0

Define BPP^0 as the class of problems with such **constant-cost** public-coin randomized protocols.

- most extreme case
- more "fine-grained" understanding
  - distinguishes public vs. private randomness
  - "dimension-free" relation [HHH22]
- ...

but structure of problems still unclear

e.g., size of largest monochromatic rectangle

The Class BPP^0

Define BPP^0 as the class of problems with such **constant-cost** public-coin randomized protocols.

- **How to use randomness to get the extreme efficient protocols?**

The Class BPP^0

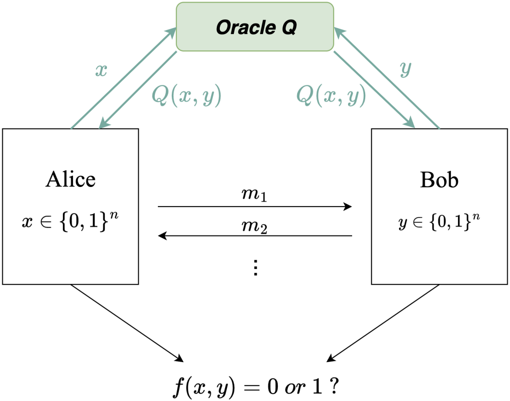
Define BPP^0 as the class of problems with such **constant-cost** public-coin randomized protocols.

- **How to use randomness to get the extreme efficient protocols?**
- **Is there a problem $\mathcal{P}$ captures all constant-cost randomized protocols**?

  $\Rightarrow$ a *complete* problem $\mathcal{P}$ for the class

Communication Complexity

Deterministic model with **oracle** access

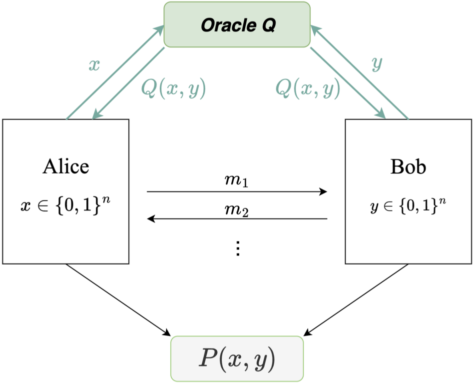


Communication Complexity

Deterministic model with **oracle** access

$\mathcal{P}$ *constant-cost reduces* to $\mathcal{Q}$ if $\mathsf{D}^{\mathcal{Q}}(\mathcal{P}) = O(1)$.

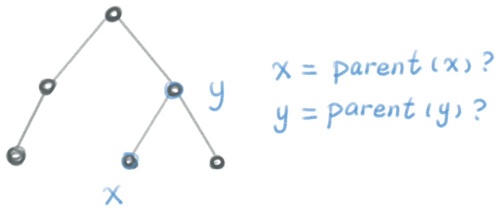
$\underbrace{\text{\# queries} + \text{\# bits exchanged}}_{O(1)}$



Constant-Cost Reduction

Example: Planar Adjacency

First suppose Alice and Bob have vertices x, y in a tree T.



Any planar graph can be partitioned into 3 forests.

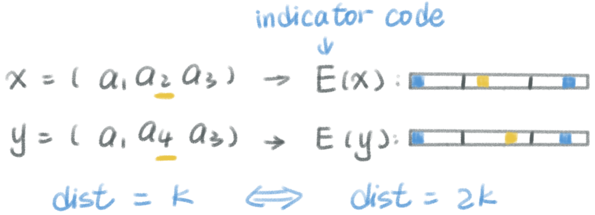
$\Rightarrow$ Adjacency in planar graphs *constant-cost reduces* to $\textsc{Eq}$.

Constant-Cost Reduction

Example: Large-Alphabet Hamming Distance

Alice and Bob receive $x, y \in [\Sigma]^n$, where $\Sigma = \{a_1, \ldots, a_m\}$.

Wish to decide whether the Σ-quary Hamming distance is k.



$\Rightarrow$ can be solved by a single query to $2k$-Hamming Distance

The Class BPP^0

- **Is there a complete problem $\mathcal{P}$ for the class under constant-cost reductions**?

  a problem that captures constant-cost randomized protocols.

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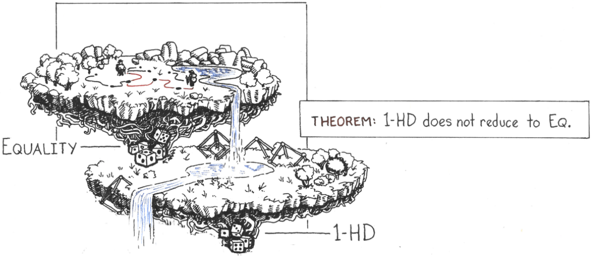
  a problem that captures constant-cost randomized protocols.

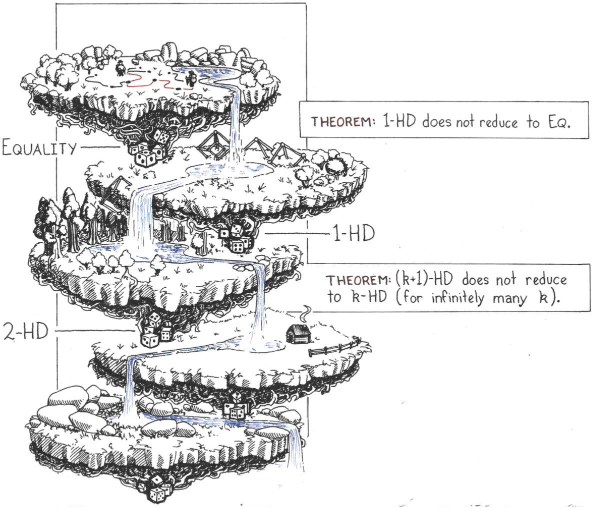
[CLV19] EQ is not complete for BPP.

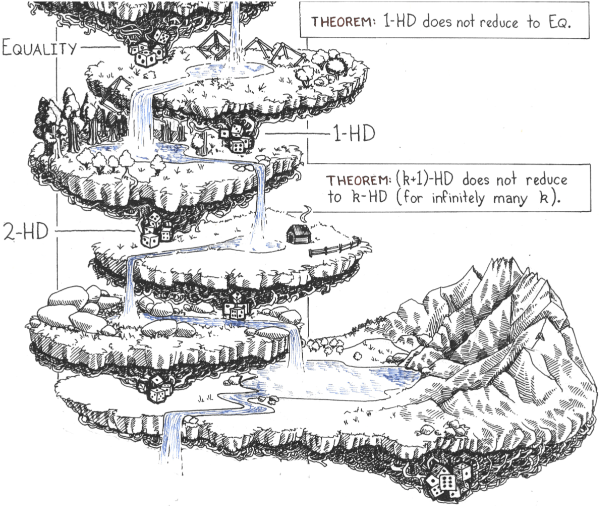
[HWZ22, HHH22] EQ is not complete for BPP^0.

[FHHH24] There is no complete problem for BPP^0.



Check out Nathan's homepage **niharms.github.io** for more art work!


THEOREM: 1-HD does not reduce to EQ.
EQUALITY
1-HD
THEOREM: (k+1)-HD does not reduce to k-HD (for infinitely many k).
2-HD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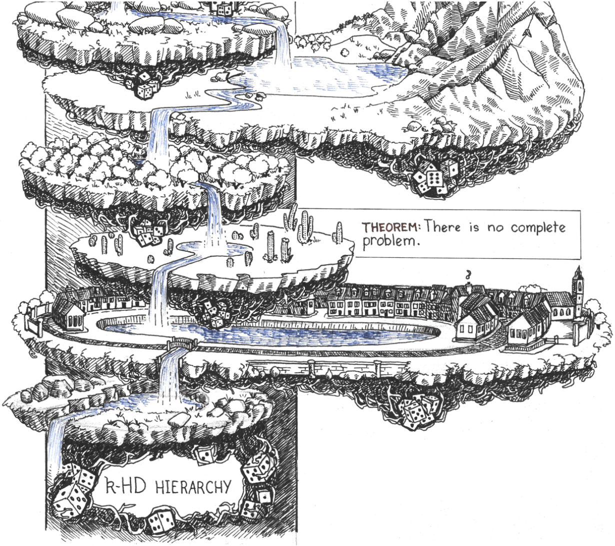

THEOREM: There is no complete problem.
k-HD HIERARCHY

Proof Overview

There is no complete problem for BPP^0

Main Theorem

For every problem $Q \in \mathrm{BPP}^0$, there is a sufficiently large k, such that HD_k does not reduce to Q.

- Requires lower bound against **arbitrary** oracles in BPP^0

Proof Overview

Main Theorem

For every problem $Q \in \mathrm{BPP}^0$, there is a sufficiently large k, such that HD_k does not reduce to Q.

- Requires lower bound against **arbitrary** oracles in BPP^0

Step 1. Constant-cost problems forbid large $\mathrm{GREATER\text{-}THAN}$ submatrices

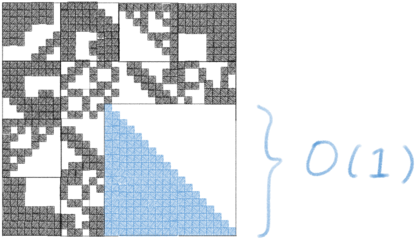
Step 2. Permutation-Invariance of HD_k and Oracle Queries

Step 3. Transform the task to lower bound against a *single* query

Step 1: Forbidden large GREATER-THAN

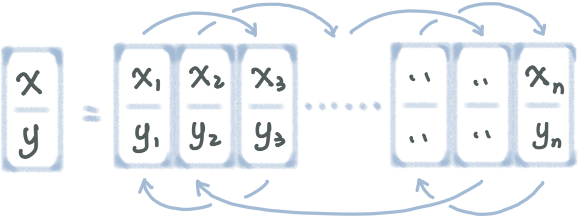
Super-constant problem: $\mathrm{R}(\mathrm{GT}) = \Theta(\log n)$.

Oracles in BPP^0 can only have constant size GREATER-THAN.



Step 2: Permutation-invariant Queries

Observe that HD_k are permutation invariant.



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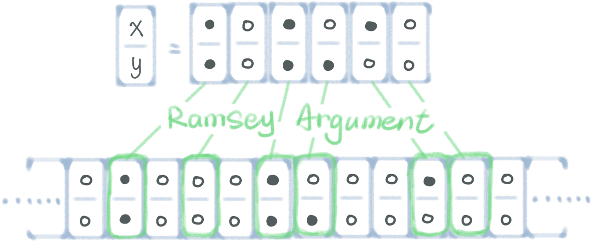
Main Lemma

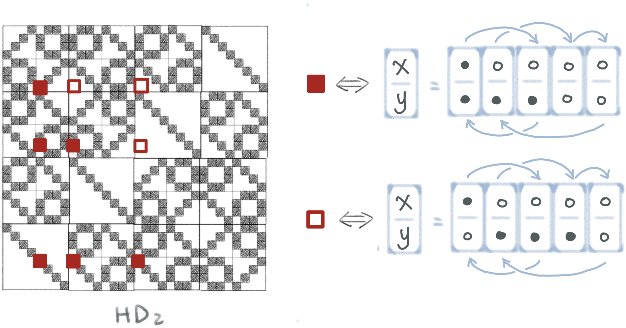
Suppose HD_k reduces to some problem $\mathcal{Q} \in \mathrm{BPP}^0$, the answers to oracle queries can be forced to be permutation invariant.



Step 3: Constant number of queries to one query

HD_k contains $\mathrm{GREATER\text{-}THAN}$ submatrix such that

- size $\geq k \times k$
- (x, y) pairs of $\mathbf{0}$-entries are permutation of each other
- (x, y) pairs of $\mathbf{1}$-entries are permutation of each other

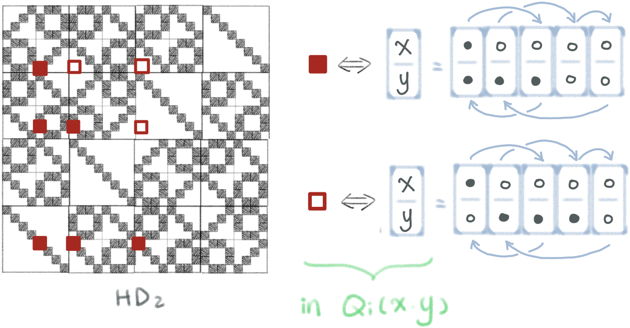


Step 3: Constant number of queries to one query

If queries $Q_1, \dots, Q_c$ are permutation-invariant:

$\Rightarrow$ exists **one** query Q_i that distinguishes 0 and 1s in the $\mathrm{GREATER\text{-}THAN}$ (of size $\geq k \times k$)

$\Rightarrow$ violate constant-cost!



Open Problems

- Does the k-**Hamming Distance hierarchy** capture all constant-cost randomized protocols?

  Solved in follow-up work - *no*

- Quantitative bounds for k-Hamming Distance separation
- Structure of problems in BPP^0
  - e.g., size of monochromatic rectangle
- Relation to other classes:
  - BPP^0 vs. Sign-rank
  - One- vs. Two-sided Error

Thank you!